RAMAKRISHNA MISSION VIDYAMANDIRA

(Residential Autonomous College affiliated to University of Calcutta)

M.A./M.Sc. SECOND SEMESTER EXAMINATION, MAY 2019

FIRST YEAR [BATCH 2018-20]

MATHEMATICS

Date : 04/05/2019

Time : 11 am – 1 pm

Paper : VI (Abstract Algebra II)

Full Marks : 50

(Answer as many questions as you possibly can. Maximum you can score is 50.)

1. a) Prove that any two composition series of a group are equivalent. (5)

   b) Prove that a group G is solvable if and only if $G/Z(G)$ is solvable. (3)

   c) Let $H \neq \{e\}$ be a subgroup of a solvable group. Prove that the derived subgroup of H is not equal to H. (2)

2. a) Show that direct product of two nilpotent groups is again nilpotent. Is the converse true? Justify your answer. (4+2)

   b) Write all abelian groups of order $2^3 \times 3^4$ upto isomorphism. (2)

   c) Show that any p-group is solvable. (2)

3. a) Let F/K be a field extension and $c \in F$. Prove that c is algebraic over K if and only if c is a root of some unique irreducible monic polynomial p(x) over K. (5)

   b) Let F/K be a finite extension. Prove that F is algebraic over K. (2)

   c) Write all intermediate fields of $\mathbb{Q}\left[\sqrt{3}+\sqrt{7}\right]$ over $\mathbb{Q}$. (2)

   d) Write all subfields of a field of order 3^{10}. (1)

4. a) Let F/K be a finite extension of an infinite field K. Then prove that F/K is a simple extension if and only if there are only a finite number of intermediate fields of F/K. (5)

   b) For any prime p, prove that there exists a field extension $F/\mathbb{Z}_p$ of arbitrary finite degree n. (3)

   c) Prove that finite fields are not algebraically closed. (2)

5. a) Let H be a finite group of automorphisms of a field F. Prove that $H = G\left(F/F_H\right)$. (6)

   b) Let $w \in \mathbb{C}$ be a primitive n-th root of unity over $\mathbb{Q}$. Then $G\left(\mathbb{Q}(w)/\mathbb{Q}\right) \simeq U_n$. (3)

   c) Give an example of a polynomial whose Galois group is S_3. (1)

6. a) Prove that n-th cyclotomic polynomial over $\mathbb{Q}$ is irreducible. (5)

b) Let K be a field of characteristic 0. Let F/K be a normal radical extension with root tower $K = K_0 \subseteq K_1 \subseteq \ldots \subseteq K_m = F$ such that $K_i = K_{i-1}(r_i)$ where r_i is a root of $x^{n_i} - a_i$, $a_i \in k_{i-1}$ for some n_i. Let $n = n_1, n_2, \ldots, n_m$. Suppose K contains all roots of unity. Then prove that $G\left(F/K\right)$ is a solvable group. (3)

c) Let K be a field of characteristic 0 and n be a positive integer. Let w be a primitive n-th root of unity. Show that $G\left(K(w)/K\right)$ is commutative. (2)

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